

Solution of a statistical mechanics model for pulse formation in lasersOmri Gat,^{*} Ariel Gordon,[†] and Baruch Fischer[‡]*Department of Electrical Engineering, Technion, Haifa 32000, Israel*

(Received 10 November 2003; revised manuscript received 13 May 2004; published 19 October 2004)

We present a rigorous statistical-mechanics theory of nonlinear many mode laser systems. An important example is the passively mode-locked laser that promotes pulse operation when a saturable absorber is placed in the cavity. It was shown by Gordon and Fischer [Phys. Rev. Lett. **89**, 103901 (2002)] that pulse formation is a first-order phase transition of spontaneous ordering of modes in an effective “thermodynamic” system, in which intracavity noise level is the effective temperature. In this paper we present a rigorous solution of a model of passive mode locking. We show that the thermodynamics depends on a single parameter, and calculate exactly the mode-locking point. We find the phase diagram and calculate statistical quantities, including the dependence of the intracavity power on the gain saturation function, and finite size corrections near the transition point. We show that the thermodynamics is independent of the gain saturation mechanism and that it is correctly reproduced by a mean field calculation. The outcome is a new solvable statistical mechanics system with an unstable self-interaction accompanied by a natural global power constraint, and an exact description of an important many mode laser system.

DOI: 10.1103/PhysRevE.70.046108

PACS number(s): 05.70.Fh, 42.55.Ah, 42.65.-k

I. INTRODUCTION

Lasers can produce light in continuous wave (cw) or pulsed manners. A special pulsed operation is mode-locking, found shortly after the laser discovery in the early 1960s [1]. Since then, mode-locked lasers became a leading way to produce ultra short pulses reaching today a few femtoseconds, or about two light-wave cycles. In a mode-locked operation, many axial modes in a broad frequency bandwidth are phase locked and thus provide one or multiple pulses in the cavity, giving at the output a light pulse train. The understanding of the conditions under which a laser operates in pulsed regime rather than in continuous wave regime is a question of great interest, both theoretical and practical. This question has been addressed in various studies, being referred to as “the second threshold” (the first one being the lasing itself) in the earlier years [2–4], and recently, in the context of laser with a saturable absorber, as the “self-starting” problem [5–12].

Formation of pulses in lasers relies on the interaction between axial modes. Such an interaction can be provided either by rendering the system time dependent (modulating) or by a suitable nonlinearity in the dynamics of the system. These two methods are commonly referred to as “active” and “passive” mode-locking, respectively. One type of nonlinearity known to encourage pulsed operation is saturable absorption. The light transmissivity through a (fast) saturable absorber is an *increasing* function of the (instantaneous) input intensity. The saturable absorber destabilizes the laser operation into configurations where most of the power is concentrated in short pulses. In the frequency or mode domain the saturable absorber induces a nonlinear four-wave-mixing in-

teraction between the modes, as does the Kerr effect, with the difference that it is dissipative rather than dispersive.

The dynamics of a laser is always subject to noise. Beside the usual noise sources present in every physical system, there is the inevitable fundamental noise of spontaneous emission. This noise is inherent in lasers, since it always accompanies coherent amplification, on which lasers rely. Therefore a model of a laser that does not take noise properly into account risks missing key features in the physics of a laser system.

The majority of laser theories treat noise as a perturbation, if at all, expecting it to manifest itself as fluctuations in the laser output. However, this approach greatly underestimates the effect of noise. It has been recently shown [13] that even very weak noise (compared to the intracavity power of the laser) is sufficient, for example, to destabilize a passively mode-locked laser, revealing a dramatic nonperturbative effect of the noise.

We have recently developed [14,15] a new approach for the many interacting mode system, with specific emphasis on aspects of pulse formation in mode-locked lasers. We established an analogy between the behavior of the electromagnetic field (the mode system in a laser) in the presence of noise and equilibrium statistical mechanics, and applied the powerful tools of statistical mechanics to the problem of mode locking. In particular it was found that the entropy associated with the noise is an essential ingredient in the theory of mode locking. This approach gave an inherent explanation for many experimental phenomena of mode-locked lasers, especially the existence of a threshold and the abruptness of formation of pulses. Passive mode locking was identified with a first-order phase transition in the model statistical mechanics system. Many other theoretical and experimental features [16] were found, among them, hysteresis, superheating and supercooling, successive formation of multiple pulses in the cavity and more.

^{*}Electronic address: omri@physics.technion.ac.il[†]Electronic address: gariel@tx.technion.ac.il[‡]Electronic address: fischer@ee.technion.ac.il

Reference [14] introduced a theoretical model of a passively mode locked laser with simplified spectral filtering of the gain, where the laser modes are restricted to a predefined band, and studied it with mean field analysis of the mode interaction induced by the saturable absorber and with numerical simulations. These indicated a first order phase transition when the effective temperature, i.e., the noise power, or alternatively the intracavity power, is varied. The ordered phase corresponds to a mode locked configuration, while the disordered phase corresponds to multimode continuous wave operation.

The present paper re-examines the model of Ref. [14], slightly modified to make it local in real space, which amounts, as we show here, to a coarse grained description of the electric field in the laser cavity. We present three new main results, which together furnish a thorough analysis of the theoretical model, showing that it captures the correct phenomenology of passive mode locking in terms of the thermodynamic picture put forward in Ref. [14].

The first main result, which is the subject of Sec. III, is an improved mean field theory, which overcomes some of the drawbacks in the analysis of Ref. [14]. The free energy in the new mean field theory is, unlike previous results, *exact* in the thermodynamic limit, where the number of active modes tends to infinity. The main advantages of the mean field analysis are its simplicity, and its direct description in terms of physical processes. Still, it relies on uncontrolled approximations based on heuristic arguments.

The second main result is an exact rigorous transfer matrix calculation of the free energy and other thermodynamic quantities in inverse powers of the number of active modes. This is the subject of Sec. IV. In addition to providing a firm footing to the arguments of Sec. III, the transfer matrix calculation provides results which are valid for a large but finite number of modes, and are important for comparison with experiments.

In Sec. V we tackle the subject of gain saturation. It is well known that the total intracavity power is determined by the saturable gain of the amplifier [19]. This fact led [14] to assume that the intracavity power is fixed once and for all, and this is the approach taken in Secs. III and IV here. However, the intracavity power can and does depend on the working conditions of the laser.

The third main result of this paper is to show that there is no essential loss of generality in this approach: The thermodynamics, and in particular the question of mode locking, depends directly only on the intracavity power and not on the details of the gain saturation mechanism. This broadly applicable result follows from the statistical physics principle of equivalence of ensembles, one with fixed power, and one with variable power, which may be likened to the canonical and grand-canonical ensembles of statistical mechanics, respectively. The results of Sec. V also allow for the actual calculation of the intracavity power for specific models of gain saturation, again a useful result for the comparison with and design of experiments.

An important corollary of all these results, which is not self-evident, is that mode locking and pulse properties depend on a *single* dimensionless parameter, the strength of the saturable absorption multiplied by the intracavity power

squared, and divided by the noise strength, and the transition occurs when this parameter crosses a threshold value, which we calculate explicitly.

As a statistical mechanics problem, the model can be likened to a gas of (complex) spins, with a $|\psi|^4$ self-interaction, and a global constraint of total amplitude, quite similar to the constraint imposed on spins in the Berlin-Kac spherical model [20]. Normally the statistical mechanics of such systems leads to simple equipartition. Here, however, the energy of the self-interaction term is *negative*, and at small enough temperature, or high enough power the instability stemming from the self-interaction drives the system into a pulsed state, where most of the power resides in a single spin. In Fourier (mode or wave number k) representation the model is equivalent to a classical complex spin chain with a special nonlocal interaction that drives the mode-locking transition, which in this representation is a standard ordering transition. Figure 1 shows the difference between typical mode-locked and non-mode-locked configurations in Fourier space and in real space in the context of the coarse grained model discussed below.

While in this work the quartic self-interaction is attributed to saturable absorption, the model can be put into a broader context. The quartic interaction is of general interest, being the lowest-order nonlinearity which is translation, inversion and rotation invariant, and local. This interaction itself has been very extensively studied, but we are not aware of previous statistical-mechanics studies of its interplay with a nonlocal power constraint. In some previous well-known statistical-mechanics studies of lasers [3], this nonlocal constraint did not exist in the model, which may explain why the mode-locking noise-induced threshold behavior has not been previously found. Since a global limitation of power is common in physical systems, our model may be an important prototype in nonlinear optics and elsewhere.

II. THE MODELING OF PASSIVE MODE LOCKING

A. The master equation

In a nondispersive lossless cavity the electric field at any given point is periodic in time, with the roundtrip time as the period. Once the laser amplifier, or other elements such as dispersion and nonlinearities, are introduced, the time dependence of the electric field is more complicated. Nevertheless, the change in the electric field between instances separated by the round-trip time is usually small. Therefore, the electric field is often described by a master equation [17], which is an equation for the slow evolution of the electric field at an arbitrary reference point inside the cavity between consecutive round-trip periods.

An alternative formulation of the master equation, which is mathematically equivalent, is obtained by passing to a frame moving with the group velocity of the optical signal. In this frame the electric field at a point x inside the cavity at time t is expressible as $E(x, t) = \text{Re}[e^{i\omega_0 t} \psi(x, t)]$, where ω_0 is the frequency of oscillations in the moving frame at the band center, and ψ is a slowly varying envelope. We assume as customary that the vector character of the electric field is not important, and treat it as a scalar. The master equation is then

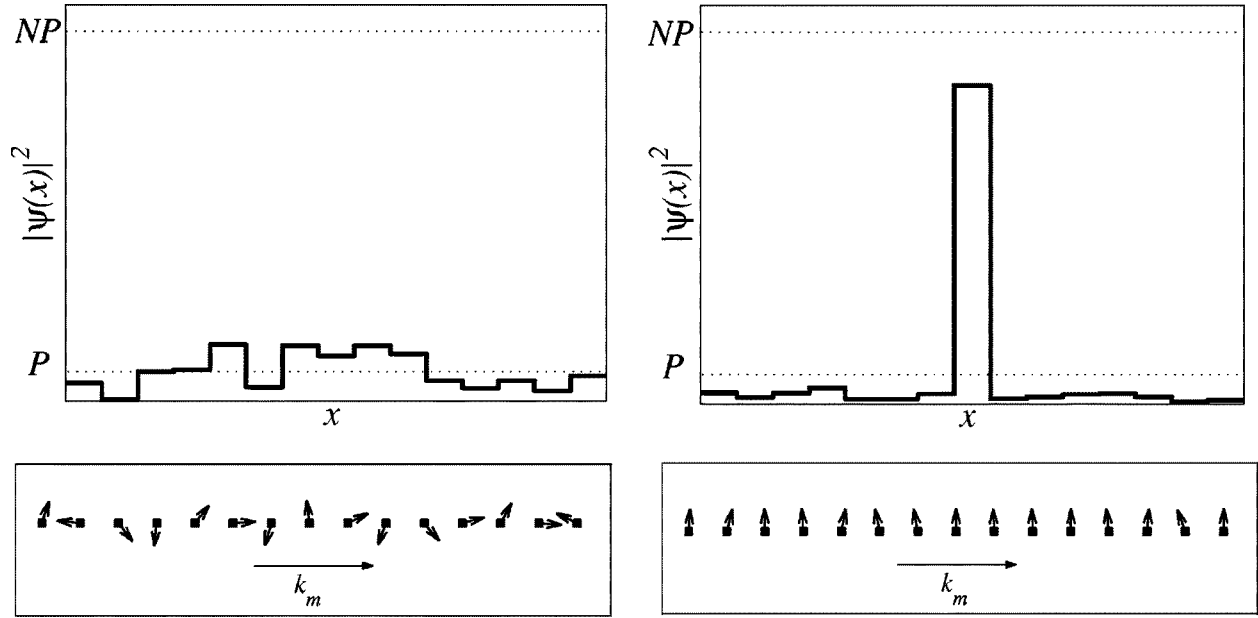


FIG. 1. The two “thermodynamic phases:” A typical non-mode-locked configuration is shown on the left-hand column of the figure and a mode-locked one on the right. The real space configurations of the coarse grained model are shown on the top row, see Eqs. (17) and (18) for definitions. The Fourier modes a_m , defined in Eq. (21), are displayed on the bottom row as a set of phasors, each arrow representing the complex value of a Fourier mode.

an equation for the temporal evolution of the envelope ψ ,

$$\partial_t \psi(x, t) = G[\psi](x, t) + \eta(x, t), \quad (1)$$

where G is the functional which represents all physical processes that modify the shape of the wave form, and η is a random term, which models the effects of noise. The cavity length denoted by L and ψ is defined on the interval $0 \leq x \leq L$ with boundary conditions $\psi(0, t) = \psi(L, t)$.

In the context of passive mode locking, the net gain has three essential components: $G = G_{\text{gain}} + G_{\text{sa}} + G_{\text{sp}}$, stemming from the actions of the saturable gain of the amplifier, the saturable absorber, and the spectral filtering of the amplifier, respectively. The net pumping of energy by stimulated emission is modeled by

$$G_{\text{gain}}(x, t) = g(\mathcal{P})\psi(x, t), \quad (2)$$

where $\mathcal{P} = (1/L) \int_0^L |\psi(x, t)|^2 dx$ is the total power in the cavity. The *saturable gain* function g is monotonically decreasing with positive values for small \mathcal{P} and negative for large \mathcal{P} , where various losses overcome the power supplied by the amplifier. It includes losses caused by the saturable absorber at zero power.

A necessary ingredient for passive mode locking is saturable absorption, wherein the dissipation losses *decrease* as the power increases. Unlike the saturable amplifier, the response of the saturable absorber is fast, so it depends on the instantaneous power $|\psi|^2$ rather than on the total power \mathcal{P} . We choose the specific form of saturable absorption

$$G_{\text{sa}}(x, t) = \gamma_s |\psi(x, t)|^2 \psi(x, t), \quad (3)$$

with $\gamma_s > 0$, valid for $|\psi|^2$ not too large [17]. Note that since losses are already taken into account in G_{gain} , the saturable

absorption term describes an additional “gain” term which is absent when $\psi=0$.

The third part models the spectral dependence of the amplifier that has a characteristic frequency, which we assume lies at the center of the wave packet. Define the Fourier expansion of the wave packet,

$$\psi(x, t) = \sum_{m=-\infty}^{\infty} b_m(t) e^{2\pi i m x / L}. \quad (4)$$

Near the resonance, the pumping efficiency falls down quadratically in the spectral distance. The net gain for modes off the center of the band is therefore reduced by

$$G_{\text{sp}}(m, t) = -\gamma_g \left(\frac{2\pi m}{L} \right)^2 b_m(t), \quad (5)$$

where γ_g is positive.

The resulting gain functional can be expressed as the functional derivative

$$G(x, t) = -\delta H[\psi] / \delta \psi^*(x) \quad (6)$$

of

$$\begin{aligned} H[\psi] = & \int_0^L dx \left(-\frac{\gamma_s}{2} |\psi(x)|^4 + \gamma_g |\psi'(x)|^2 \right) \\ & + LU \left(\frac{1}{L} \int_0^L |\psi(x)|^2 dx \right), \end{aligned} \quad (7)$$

with the definition $U'(\mathcal{P}) = -g(\mathcal{P})$.

In addition to the deterministic gain terms, the laser light is also subject to the effects of the random noise η , which turn out to be crucial in the context of passive mode locking.

An inherent source of noise is the spontaneous emission from the amplifier. This type of noise is well modeled as white and Gaussian. We will suppose therefore that η is a complex Gaussian white uncorrelated noise with a covariance function,

$$\langle \eta^*(x, t) \eta(x', t') \rangle = 2TL \delta(x - x') \delta(t - t'), \quad (8)$$

where $\langle \cdot \rangle$ stands for an ensemble average.

B. The invariant measure

It is well known [14,21] that the invariant measure of gradient flow with additive white noise, such as Eq. (1) with Eqs. (6) and (8) is

$$\rho[\psi] = Z^{-1} e^{-H[\psi]/(LT)}, \quad (9)$$

where the partition function is

$$Z = \int [d\psi][d\psi^*] e^{-H[\psi]/(LT)}, \quad (10)$$

and the notation $[d\psi]$ indicates functional integration. Thus, the study of passive mode locking in a laser with spectral filtering and saturable absorption reduces to the analysis of the statistical mechanics system described by Z .

The physical ingredients in the gain functional discussed so far do not include refractive effects such as dispersion and the Kerr nonlinearity, which are important in most laser systems [17]. When such terms are included in G , it can no longer be written as a gradient as in Eq. (6), and the invariant measure is, in general, much more complicated. Nevertheless, as pointed out in a previous work [15], ρ serves as the invariant measure also when dispersive effects are included, provided that a certain integrability condition, sometimes called the soliton condition, holds. Furthermore, the numerical studies of Ref. [15] indicate that many of the qualitative properties that results derived in the integrable case, where the invariant measure is given by Eq. (9), persist even when the integrability condition ceases to hold. The purpose of this work is to study in detail the integrable case.

C. The coarse-grained model

We proceed to present a modified model, first suggested in Ref. [14], characterized by a simplified spectral profile. Namely, we replace the quadratic filtering of Eq. (5), by a spectrum limited to a finite band, in which the gain acts equally on all modes. It is based on the following observation. The spectral filtering term $\gamma_g |\psi'(x)|^2$ in H introduces correlations between the electric field in neighboring positions in the cavity, counteracting the tendency of the saturable absorption to concentrate the power in an increasingly thinner interval; the spectral filtering introduces a length scale over which the electric field is smooth. The essence of this behavior is captured by assuming that the electric field envelope ψ is constant on an interval of size comparable with the correlation length, while neglecting all correlations between ψ in different intervals. The function ψ is then represented on each interval by a single (complex) degree of freedom (see Fig. 1).

Taking N such intervals we find that the functional H is replaced by

$$\tilde{H}_N(\psi_1, \dots, \psi_N) = -\frac{\gamma_s L}{2N} \sum_{n=1}^N |\psi_n|^4 - LU(\mathcal{P}) \quad (11)$$

and the intracavity power is expressed by

$$\mathcal{P} = \frac{1}{N} \sum_n |\psi_n|^2. \quad (12)$$

Furthermore, the conclusions of Sec. II B continue to hold with H replaced by \tilde{H} , and the partition function becomes

$$\tilde{Z}_N = \int \prod_n \frac{d\psi_n d\psi_n^*}{2\pi} e^{-\tilde{H}_N[\psi]/(LT)}. \quad (13)$$

The statistical mechanics problem defined by Eqs. (11)–(13) is the main object of study in this paper. At zero temperature, which corresponds to noiseless dynamics, the quartic term in \tilde{H} pushes all the available power into a single degree of freedom (whose identity depends on the initial conditions). This is the mode-locked state. In the opposite situation of high T or $\gamma_s = 0$ the power is randomly distributed, and mode locking is absent.

While the coarse-grained model described in this section is admittedly somewhat artificial, it is an important object of study as it provides the simplest example of a system in which the phenomenon of passive mode-locking *transition* occurs. In this sense it plays in the theory of passively mode-locked laser a role similar to the one of the Ising model in equilibrium statistical mechanics: It is a theoretical laboratory which does not quantitatively approximate any real system, but exhibits in a mathematically simple setting the correct phenomenology of actual experiments.

D. The thermodynamic limit

Our analysis relies crucially on N being very large. In the way we stated the problem, N is the ratio between the laser cavity length and the width of a pulse. In short-pulse laser this is a natural large parameter, with values ranging from 10^2 to 10^9 depending on the cavity length L and the bandwidth. Taking $N \rightarrow \infty$ and expanding in $1/N$ looks therefore promising.

In order to follow such a procedure, a model that has a well-defined $N \rightarrow \infty$ limit is required. In particular, thermodynamic averages of physical quantities should have a finite $N \rightarrow \infty$ limit. It is not difficult to see, however, that this does not hold in the statistical mechanics system of Eqs. (11)–(13). For example, the $T=0$ value of the intracavity power \mathcal{P} is the minimum of

$$-\frac{1}{2} \gamma_s N L \mathcal{P}^2 - LU(\mathcal{P}) \quad (14)$$

with respect to \mathcal{P} , which naively diverges with N .

In a given physical system where N is finite and fixed, this does not constitute a problem. However, here, for the purpose of theoretical analysis, we wish to approximate the large but finite N model by a *fictitious* model for which N

$\rightarrow \infty$. From Eq. (14) it is evident that this fictitious model is not simply the $N \rightarrow \infty$ limit of the model defined by Eqs. (11)–(13).

If in Eq. (11) and hence also in Eq. (14) we make the replacement

$$U(\mathcal{P}) \rightarrow NTu(\mathcal{P}), \quad (15)$$

we obtain a model where \mathcal{P} , and other quantities as explained below, have a finite thermodynamic limit. An *actual* system with given N , $U(\mathcal{P})$, and T is well-approximated by the limit system with $u(\mathcal{P}) = U(\mathcal{P})/(NT)$.

A renormalization of the system parameters such as the one presented here is quite often necessary in problems of statistical mechanics in order to obtain finite results in the thermodynamic limit, and the meaning and practical use of renormalization in the general context is well understood [22].

However, a more fundamental difficulty remains: Due to the nature of the mode-locking transition, the ratio of the peak power, the maximal value of $|\psi_n|^2$, to the intracavity power diverges like N when N is large. This means that only one of these quantities can achieve a finite well-defined thermodynamic limit.

In the renormalization scheme of Eq. (15), which is used in the present paper (as well as in Ref. [14]), the intracavity power \mathcal{P} and the phase transition “temperature” reach a well-defined limit as $N \rightarrow \infty$, while the peak power diverges linearly in N . A natural order parameter is therefore

$$M = \left(\frac{1}{N^2} \sum_n \langle |\psi_n|^4 \rangle \right)^{1/4}, \quad (16)$$

where $\langle \rangle$ stands for expectation with respect to the invariant measure. In the ordered phase M has a finite nonzero thermodynamic limit, and in the disordered phase M tends to zero as $N \rightarrow \infty$.

E. Fixed power ensemble

Since the quartic term in the Hamiltonian \tilde{H} is unbounded from below, the gain saturation term $U(\mathcal{P})$ is essential to ensure stability, preventing the system from cascading into states with arbitrarily low \tilde{H} . This reflects the well-known fact that lasers owe their stability to gain saturation [18,19]. This is analogous to the role of the chemical potential in the grand-canonical ensemble in standard statistical mechanics, which limits the number of particles in the system.

An alternative approach is to suppose that the intracavity power \mathcal{P} has a *fixed* value P , whose analogue in textbook statistical mechanics is the canonical ensemble where the number of particles is fixed. This is the scheme used in Ref. [14]. One of the main results of the present work is an equivalence of ensembles. The thermodynamics obtained in the fixed- and variable-power ensembles are equivalent. The variable-power ensemble must be used if $\langle \mathcal{P} \rangle$ is not known. These issues are discussed in Sec. V.

In the fixed power ensemble there is no need to include the gain saturation term, and the partition function is defined by

$$Z_N(\gamma_s, T, P) = \int \prod_n \frac{d\psi_n d\psi_n^*}{2\pi} e^{-H_N[\psi]/T} \delta(\mathcal{P}[\psi] - P), \quad (17)$$

with the reduced Hamiltonian

$$H_N[\psi] = -\frac{\gamma_s}{2N} \sum_{n=1}^N |\psi_n|^4. \quad (18)$$

The change of variables $y_n = |\psi_n|^2$ in Eq. (17) leads to the simpler form

$$Z_N(\gamma_s, T, P) = \int \prod_n dy_n e^{(\gamma_s/2NT) \sum_n y_n^2} \delta\left(\frac{1}{N} \sum_{n=1}^N y_n - P\right), \quad (19)$$

where the y_n integrations are from 0 to ∞ . Another change of variables $y_n \rightarrow P y_n$ leads to the useful scaling relation

$$Z_N(\gamma_s, T, P) = P^{N-1} \mathcal{Z}_N(\gamma_s P^2/T), \quad (20)$$

where

$$\mathcal{Z}_N(\gamma) \equiv Z_N(\gamma, 1, 1).$$

An important conclusion has already been reached: The thermodynamics depends on the single parameter $\gamma \equiv \gamma_s P^2/T$. Equation (20) proves this in the fixed power scheme, while the equivalence of ensembles extends this to the general case.

In the rest of this paper we solve the statistical mechanics problem of the coarse grained mode. First, in Sec. III a mean field theory is developed for the fixed-power ensemble, which is later shown to be exact in the thermodynamic limit. Then, in Sec. IV we solve the statistical mechanics problem, still in the fixed-power ensemble, by developing a uniform asymptotic expansion of \mathcal{Z}_N in inverse powers of N . Finally, using the results of Sec. IV we calculate the partition function also in the variable-power ensemble in Sec. V, which completes the solution of the coarse-grained model.

III. MEAN FIELD THEORY

The free energy in the coarse-grained model lends itself to a mean field analysis when formulated in Fourier (mode) space. In Fourier representation mode-locking manifests as ordering of the phases of the various modes, see Fig 1. Introducing the discrete Fourier transform

$$\psi_n = \sum_{m=1}^N a_m e^{2\pi i m (n/N)} \quad (21)$$

the Hamiltonian (18) is expressed in terms of Fourier modes by

$$H[a] = -\frac{\gamma_s}{2} \sum_{m_1-m_2+m_3-m_4=pN} a_{m_1}^* a_{m_2} a_{m_3}^* a_{m_4}, \quad (22)$$

where p is an integer (whose only possible values are $p = -1, 0, 1$, since m_1, \dots, m_4 are between 1 and N). \mathcal{P} is now given by

$$\mathcal{P} = \sum_m |a_m|^2. \quad (23)$$

The main disadvantage of the Fourier space formulation is that the nonlinear term becomes complicated and nonlocal. Mean field theory overcomes this difficulty by assuming that the different modes are uncorrelated and characterized by a common probability distribution function $\rho_{\text{mf}}(a)$. When the problem is formulated in Fourier space, the mean field approximation is quite plausible, since the interaction term involves all the degrees of freedom.

In the mean field framework the free energy per degree of freedom is

$$F \equiv -\frac{\log Z}{N} = -\frac{\gamma_s}{2T} N^2 \langle |a|_{\text{mf}} \rangle^4 + \langle \log \rho \rangle_{\text{mf}}, \quad (24)$$

where $\langle \cdot \rangle_{\text{mf}}$ stands for expectation value with respect to ρ_{mf} . Gain saturation is included by demanding that

$$\langle |a|^2 \rangle_{\text{mf}} = \frac{P}{N}, \quad (25)$$

i.e., $\langle \mathcal{P} \rangle_{\text{mf}} = P$. This is not enough in itself to satisfy the constraint of fixed power, since \mathcal{P} fluctuates. However, in the thermodynamic limit, which is always necessary for the validity of the mean field approximation, the fluctuations tend to zero, and Eq. (25) is justified.

Following the standard procedure of mean field calculations [22], ρ_{mf} is found by minimizing the free energy subject to the constraint, Eq. (25). A necessary condition for the minimization of F is stationarity with respect to variations of ρ ,

$$\begin{aligned} 0 &= \frac{\delta}{\delta \rho(a)} [F + \lambda (\langle |a|^2 \rangle - P/N)] \\ &= -2 \operatorname{Re}(\gamma_s/T) N^2 \langle |a| \rangle^2 \langle a^* \rangle a + [\log \rho(a) + 1] + \lambda |a|^2, \end{aligned} \quad (26)$$

where λ is a Lagrange multiplier. The solution of Eq. (26) is a Gaussian probability distribution function

$$\rho(a) = \frac{1}{\pi \sigma^2} e^{-(|a-\langle a \rangle|^2)/\sigma^2}. \quad (27)$$

$\langle a \rangle$ and σ are related by Eq. (25) which implies

$$\langle |a|^2 \rangle_{\text{mf}} = \langle |a|_{\text{mf}} \rangle^2 + \sigma^2 = \frac{P}{N}. \quad (28)$$

ρ_{mf} is therefore characterized by the single parameter $M = \sqrt{N/P} \langle |a|_{\text{mf}} \rangle$. From Eq. (28) one can see that $0 \leq |M|^2 \leq 1$. When $M=0$ the phases of the modes are completely random, which means that in real space the power is uniformly distributed. When $M>0$ the modes are correlated which in real space means that a macroscopic fraction of the power resides in the variable ψ_N . M is therefore an order parameter, which can be shown to asymptotically coincide with the previous definition of the order parameter Eq. (16).

We note that although the present formulation of mean field theory allows only for pulse formation at a specific

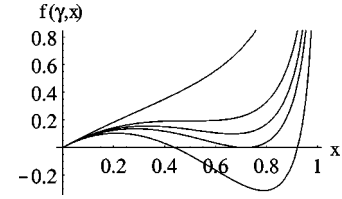


FIG. 2. The thermodynamic potential $f(\gamma, y)$ as a function of $y=M^2$ for several values of γ ranging from 2 to 6 with higher values of γ corresponding to lower curves. Curves with two (local) minima correspond to systems with a metastable state. The critical γ (γ^*) is by definition the one where the values of $f(\gamma, \cdot)$ at the two minima are equal, which implies a first order phase transition.

point in the cavity, because it assumes that the mode variables a_m are identically distributed, the mean field theory yields the correct thermodynamics. The reason is that inclusion of ordered configurations with different pulse positions would contribute a term of $O(\log N)$ to the entropy, which is negligible in the thermodynamic limit, where the entropy is $O(N)$.

We proceed to define a new thermodynamic potential $f(\gamma, y)$, where $y=M^2$ and $\gamma=\gamma_s P^2/T$, which is the free energy for a given value M . Using Eqs. (27) and (28) in Eq. (24) it is found that

$$f(\gamma, y) = -\left(\frac{\gamma}{2} y^2 + \log(1-y) \right), \quad (29)$$

up to unimportant additive terms independent of γ and M . The free energy F for a given γ is the global minimum of $f(\gamma, \cdot)$, and the abscissa of the minimum, \bar{y} , is the square of the order parameter.

The function f of Eq. (29) has a single minimum for $\gamma \leq 4$, at $y_0=0$. The vanishing of M means that the phases are not locked corresponding to the disordered, non-mode-locked phase. For $\gamma > 4$ there exists an additional (local) minimum, $y_1 > 0$, see Fig. 2, which corresponds to the mode-locked state. However, for $4 \leq \gamma \leq \gamma^* \approx 4.91$ $f(\gamma, y_1) > f(\gamma, y_0)$, which means that the mode-locked state is metastable, and the true equilibrium is still disordered, $\bar{y} = y_0$. At $\gamma = \gamma^*$ the two minima exchange stability, and for all $\gamma > \gamma^*$ the equilibrium state is mode locked, with $\sqrt{y_1}$ as the value of the order parameter. γ^* is the solution of the equation $f(\gamma, y_0) = f(\gamma, y_1(\gamma))$,

$$\frac{(\sqrt{\gamma^*} + \sqrt{\gamma^* - 4})^2}{8} = \log \frac{\sqrt{\gamma^*}(\sqrt{\gamma^*} + \sqrt{\gamma^* - 4})}{2}. \quad (30)$$

In terms of the original variables γ_s , T , and P , the phase transition point is therefore

$$\frac{\gamma_s P^2}{T} = \gamma^* \approx 4.91. \quad (31)$$

In order to compare it to the result in Ref. [14], one should remember the difference in the modeling of spectral filtering here and there. In Ref. [14] we imposed ‘‘Dirichlet’’ boundary conditions in the Fourier space, while here, the coarse graining method actually induces periodic boundary con-

sitions in Fourier space. Since the interaction in Fourier space is long ranged, this leads to a difference. The Hamiltonian in Ref. [14] was identical to Eq. (22), except that $p=0$ only. The number of quartets with $k=0$ is $2/3$ of their number with $p=-1,0,1$. In the mean field approximation this would simply lead to a factor of $3/2$ in the transition temperature,

$$\frac{\gamma_s P^2}{T} = \frac{3}{2} \gamma^* \approx 7.4. \quad (32)$$

This is very close to the result of $\gamma_s P^2/T \approx 7.7$ obtained in Ref. [14]. (We note that the mode-locking transition was specified in Ref. [14] in terms of $1/\gamma$.) The mean field theory presented here is better, since here we do not make the ansatz of separating the modulus and angle in ρ_{mf} . Here we also end up with the simple analytical expression (29). Since the Hamiltonian (22) is translation invariant in Fourier space, mean field theory is more suitable for its analysis than for that of its counterpart from Ref. [14].

In summary, it has been demonstrated in the mean field context that the mode locking transition is a standard first order transition, accompanied by coexistence and metastable configurations in its neighborhood. The mean field theory involves an uncontrolled approximation, which is hard to justify rigorously. In the present work the justification will ultimately follow from the real space analysis given in the next section.

Evidently, the transition condition depends, although not greatly, on the spectral filtering scheme. Analysis of the parabolic filtering scheme, which we do not pursue here, yields yet another value for the transition temperature, close to (31) and (32).

IV. FIXED POWER FINITE N ANALYSIS

In this section we calculate an asymptotic expansion of the partition function $Z_N(\gamma, P)$ in decreasing powers of N , the number of degrees of freedom. All information pertaining to the passive mode-locking transition is then obtainable in a standard manner. In particular, we show that the mean field calculations give the exact free energy.

Our starting point is a recursive version of Eq. (19) for Z_N , obtained by performing only $N-1$ of the y integrations,

$$Z_N(\gamma_s, T, P) = \int dy_N e^{(\gamma_s/2NT)y_N^2} Z_{N-1} \times \left(\frac{N-1}{N} \gamma_s, T, \frac{N-y_N}{N-1} P \right). \quad (33)$$

After using the scaling relation Eq. (20) and making a change of the integration variable we obtain a recursive equation for \mathcal{Z} ,

$$Z_N(\gamma) = N \left(\frac{N}{N-1} \right)^{N-1} \int_0^1 dy e^{(\gamma/2)Ny^2} \times (1-y)^{N-2} Z_{N-1} \left(\gamma \frac{N}{N-1} (1-y)^2 \right). \quad (34)$$

This is the fundamental equation of the real space analysis.

We will show that when N is large the only significant contribution to the y integration in Eq. (34) comes from the vicinity of one or two values of y which maximize the integrand, one of which is $y_0=0$. The integration on other parts of the interval is exponentially small in N and will be neglected. The case of a single maximizing point will be shown to correspond to invariant measures concentrated on configurations where the amplitude of all degrees of freedom is $O(1)$, i.e., non-mode-locked, disordered configurations. This happens for small enough γ . When there are two maximizing points the typical configurations are such that a finite fraction of the power is concentrated in a *single* degree of freedom, while the amplitude of other degrees of freedom is again $O(1)$. These mode-locked configurations arise for large enough values of γ . We show that these are the only two possibilities.

A. The disordered phase

We first tackle the case of small γ . To this end we use the Fourier representation of the delta function in Eq. (19) to re-express \mathcal{Z} by

$$\mathcal{Z}_N(\gamma) = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} e^{-z} \left(\int_0^{C_N} dy e^{(\gamma/2N)y^2 + (1/N)zy} \right)^N. \quad (35)$$

for some $C_N \geq N$. As long as $C_N \geq N$, the right-hand side of Eq. (35) is independent of C_N . One can now expand the quadratic term in the exponential in a Taylor series keeping the first two terms, carry out the y integration, and then take $C_N \rightarrow \infty$ giving

$$\mathcal{Z}_N(\gamma) \sim \int \frac{dz}{2\pi i} e^{-z} \left(-\frac{N}{z} - \frac{\gamma N^2}{z^3} \right)^N. \quad (36)$$

The contour of integration must be deformed so as to avoid the singularity at $z=0$. A standard argument shows that the contour should be moved to the left so that it crosses the real line at a negative value. The integral (36) can then be calculated by pushing the contour through the singularity at $z=0$ and then to $\text{Re } z = \infty$. The exponential in the integrand makes the integration at infinity vanish in the limit, leaving only integration on a contour surrounding $z=0$ clockwise. Using Cauchy's theorem this evaluates to

$$\begin{aligned} \mathcal{Z}_N(\gamma) &\sim N^N \oint \frac{dz}{2\pi i} e^{-z} \sum_n \binom{N}{n} \frac{(\gamma N)^n}{(-z)^{N+2n}} \\ &= \sum_n \frac{\gamma^n N^{N+n} N!}{n! (N-n)! (N+2n)!}, \end{aligned} \quad (37)$$

or, using Stirling's formula,

$$\mathcal{Z}_N(\gamma) \sim \frac{e^N}{\sqrt{2\pi N}} \sum_n \frac{\gamma^n}{n!} = e^\gamma \frac{e^N}{\sqrt{2\pi N}} \equiv \mathcal{Z}_N^{(0)}(\gamma). \quad (38)$$

$\mathcal{Z}_N^{(0)}$ certainly provides an asymptotic approximation of $\mathcal{Z}_N(\gamma)$ as $\gamma \rightarrow 0$, but we shall show that it also serves as the leading term of $\mathcal{Z}_N(\gamma)$ as $N \rightarrow \infty$ for all $0 \leq \gamma < \gamma^*$. This is achieved by showing that the recursive equation (34),

$$\begin{aligned} \mathcal{Z}_N^{(0)}(\gamma) &\sim Ne \int dy e^{(\gamma/2)Ny^2} (1-y)^{N-2} \mathcal{Z}_{N-1}^{(0)}(\gamma(1-y)^2) \\ &= N \frac{e^N}{\sqrt{2\pi N}} \int_0^\infty dy \frac{e^{\gamma(1-y)^2}}{(1-y)^2} e^{N[(\gamma/2)y^2 + \log(1-y)]} \end{aligned} \quad (39)$$

holds for γ in this interval. Here and below the symbol \sim stands for asymptotic for large N . Consider the last integral. When $N \rightarrow \infty$ the integrand becomes strongly peaked near the global minimum $\bar{y}(\gamma)$ of $f(\gamma, y) = -[(\gamma/2)y^2 + \log(1-y)]$. The function f is precisely the thermodynamic potential encountered in the context of the mean field approximation, see Eq. (29). As shown above (Sec. III), $\bar{y}(\gamma) = y_0 \equiv 0$ for $\gamma < \gamma^*$. For these values of γ only the neighborhood of $y=0$ must be taken into account and the integral on the right-hand side of Eq. (39) is approximately

$$Ne^\gamma \frac{e^N}{\sqrt{2\pi N}} \int_0^1 dy (1-y)^{N-2} \sim e^\gamma \frac{e^N}{\sqrt{2\pi N}} \sim \mathcal{Z}_N^{(0)}, \quad (40)$$

which establishes

$$\mathcal{Z}_N(\gamma) \sim \mathcal{Z}_N^{(0)}(\gamma), \quad \gamma < \gamma^*. \quad (41)$$

The thermodynamics now follow straightforwardly. For example, the free energy per degree of freedom is $F(\gamma) = (1/N) \log \mathcal{Z}_N \sim 1$, independent of γ to leading order, and the expectation value of $|\psi|^4$ in the invariant measure is

$$\langle |\psi|^4 \rangle = 2 \frac{\mathcal{Z}'_N(\gamma)}{\mathcal{Z}_N(\gamma)} \sim 2 \quad (\gamma < \gamma^*), \quad (42)$$

also independent of γ in the leading order. In particular, the order parameter M from Eq. (16) equals zero, showing that this is indeed a disordered configuration.

B. The mode-locked phase

We turn now to the case $\gamma > \gamma^*$, where $\bar{y} = y_1 > 0$. We can no longer expect that $\mathcal{Z}_N \sim \mathcal{Z}_N^{(0)}$, but the mean field calculations suggest that

$$\mathcal{Z}_N(\gamma) \sim A_N(\gamma) e^{-NF(\gamma)}, \quad (43)$$

where $F(\gamma) = f(\gamma, \bar{y}(\gamma)) - 1$ and A_N is subexponential in N . The results of the preceding section imply that the asymptotic form (43) is valid for $\gamma < \gamma^*$, since then $\bar{y} = y_0$, and $F = -1$. Using Eq. (34) for $\gamma > \gamma^*$, we presently show that Eq. (43) is valid for all $\gamma \neq \gamma^*$ and find explicit expressions for A_N .

Substituting Eq. (43) in Eq. (34) gives the asymptotic equation

$$\begin{aligned} A_N(\gamma) e^{-NF(\gamma)} &\sim Ne \int dy \frac{e^{(\gamma/2)y^2}}{1-y} A_{N-1}[\gamma(1-y)^2] \\ &\quad \times e^{-(N-1)[f(\gamma, y) + F(\gamma(1-y)^2)]}. \end{aligned} \quad (44)$$

As before, the integration is concentrated near the maximal points of the large exponential, i.e., the minima (as a function of y) of $f(\gamma, y) + F[\gamma(1-y)^2]$. Recalling the definition of F the minimization problem turns into

$$\begin{aligned} \min_y f(\gamma, y) + F[\gamma(1-y)^2] &= \min_{y, \tilde{w}} -\gamma y^2 - \log(1-y) - \gamma(1-y)^2 \tilde{w}^2 - \log(1-\tilde{w}) \\ &= \min_{y, w} -\frac{\gamma}{2}(y^2 + w^2) + \log(1-y) - w, \end{aligned} \quad (45)$$

where we have set $w = \tilde{w}(1-y)$. It is straightforward to check that either y or w must vanish at the minimum. For $\gamma > \gamma^*$ there are two possibilities, $y = y_1(\gamma), w = 0$, and $y = 0, w = y_1(\gamma)$, and the minimal value is the same in both cases.

The conclusion is that the integration receives two main contributions, one from the neighborhood of $y=0$, and one from the neighborhood of $y=\bar{y}$, which we denote by I_0 and I_1 , respectively. I_0 is found by approximating the exponential near $y=0$ and evaluating prefactors at $y=0$ giving

$$\begin{aligned} I_0 &\sim Ne A_{N-1}(\gamma) e^{-(N-1)F(\gamma)} \int_0^1 dy (1-y)^{N-2} e^{2(N-1)F'(\gamma)y} \\ &\sim \frac{e^{F(\gamma)+1}}{1-2F'_1(\gamma)} A_N(\gamma) e^{-NF(\gamma)}, \end{aligned} \quad (46)$$

the assumption that A_N is subexponential in N was used to approximate $A_{N-1} \sim A_N$.

For the calculation of I_1 we need to evaluate F near $\chi[1 - \bar{y}(\gamma)]^2$. It follows from the properties of f that this is always strictly less than γ^* , where $F \equiv -1$. Therefore

$$\begin{aligned} I_1 &\sim \frac{\sqrt{N} e^{\gamma(1-\bar{y})^2}}{\sqrt{2\pi}(1-\bar{y})^2} e^{NF(\gamma)} \int_{-\infty}^{\infty} dy e^{(N-1)F^{(2)}(\gamma)[(y-\bar{y})^2/2]} \\ &\sim \frac{e^{\gamma(1-\bar{y})^2}}{(1-\bar{y})^2 \sqrt{F^{(2)}(\gamma)}} e^{-NF(\gamma)}, \end{aligned} \quad (47)$$

where $F^{(2)}(\gamma) = \partial_y^2 f(\gamma, \bar{y}(\gamma))$. Combining these results with Eq. (44), we get a linear equation for A_N whose solution is

$$\begin{aligned} A_N(\gamma) &= \frac{e^{\gamma(1-\bar{y})^2}}{(1-\bar{y})^2 \sqrt{F^{(2)}(\gamma)}} \frac{1-2F'(\gamma)}{1-2F'(\gamma) - e^{F(\gamma)+1}}, \\ &\quad \gamma > \gamma^*. \end{aligned} \quad (48)$$

This establishes Eq. (43), with explicit values for A_N for all $\gamma \neq \gamma^*$.

We find now that $F(\gamma)$ is indeed the free energy, consistently with the mean field theory. Since

$$\langle |\psi|^4 \rangle \sim 2(\log \mathcal{Z}_N)'(\gamma) = 2 \frac{A'(\gamma)}{A(\gamma)} - 2NF'(\gamma), \quad (49)$$

the order parameter $M = (2F'(\gamma))^{1/4}$ is nonzero for $\gamma > \gamma^*$, showing that mode locking occurs for such γ . Using the definition of F we can calculate explicitly

$$-2F'(\gamma) = -\frac{d}{d\gamma} f(\gamma, \bar{y}(\gamma)) = -\partial_y f(\gamma, \bar{y}(\gamma)) = \bar{y}^2, \quad (50)$$

since f is by definition stationary with respect to y at \bar{y} . This means that $M = \sqrt{\bar{y}}$, also in accordance with the mean field

calculations. It is possible to show by calculating higher moments that M^2 is the power concentrated in a *single* degree of freedom. This result, which in physical terms means that mode-locking results in a single pulse, can be traced to the fact that the integral in Eq. (44) receives contributions only from y_0 and y_1 .

C. The transition region

The analysis of the preceding section does not apply to the case where γ is precisely equal to γ^* . For example, Eq. (48) would imply that $A_N(\gamma^*)$ is infinite, since the derivative in $F'(\gamma)$ should be taken from below. More importantly, the asymptotic approximation Eq. (43) is not uniform in N near γ^* , because it neglects the contribution of the metastable state. An asymptotic approximation for \mathcal{Z} which is valid and uniform for all γ is

$$\mathcal{Z}_N(\gamma) \sim e^\gamma \frac{e^N}{\sqrt{2\pi N}} + A_N^*(\gamma) e^{-NF^*(\gamma)}, \quad (51)$$

where $F^*(\gamma) = f(\gamma, y_1(\gamma))$, and A_N^* is given by Eq. (48) replacing everywhere F by F^* and \bar{y} by y_1 . The uniform approximation is a *continuous* function of γ which reduces to the nonuniform approximation for $|\gamma - \gamma^*| \gg 1/N$.

Observables in systems with a finite number of degrees of freedom exhibit crossover behavior in the mode-locking transition, rather than the sharp, discontinuous dependence on parameters predicted in the thermodynamic limit. When the number of degrees of freedom is not too large, the crossover is measurable and describable by the uniform approximation. For example,

$$\langle |\psi|^4 \rangle \sim 2[a_N(\gamma) + Nb_N(\gamma)y_1(\gamma)^2], \quad (52)$$

where

$$a_N = \frac{e^\gamma \frac{e^N}{\sqrt{2\pi N}} + (A_N^*)'(\gamma) e^{-NF^*(\gamma)}}{e^\gamma \frac{e^N}{\sqrt{2\pi N}} + A_N^* e^{-NF^*(\gamma)}} \quad (53)$$

and

$$b_N = \frac{A_N^*(\gamma) e^{-NF^*(\gamma)}}{e^\gamma \frac{e^N}{\sqrt{2\pi N}} + A_N^* e^{-NF^*(\gamma)}}. \quad (54)$$

For moderate values of N , there is a significant interval in γ below the transition where $\langle |\psi|^4 \rangle$ is much larger than its value in the thermodynamic limit. A comparison between the uniform and nonuniform approximations to $\langle |\psi|^4 \rangle$ is shown in Fig. 3.

V. THERMODYNAMICS WITH VARIABLE TOTAL POWER

In the preceding section we developed a systematic approximation scheme for the partition function Z of the passive mode-locking model as a function of the nonlinearity

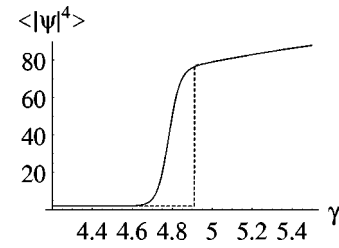


FIG. 3. A comparison between the value of the order parameter in the uniform (full line) and the nonuniform (broken line) approximation as a function of the nonlinearity parameter γ for a system with $N=150$ degrees of freedom. The nonuniform approximation breaks down near the transition point γ^* .

strength γ_s , the fixed total intracavity power P , and the temperature T . This allowed us to calculate the free energy per degree of freedom, and we found that mode locking occurs whenever $\gamma_s P^2/T$ is greater than a critical value γ^* .

However, in experimental situations the intracavity power \mathcal{P} is not fixed in advance. Rather it is a fluctuating quantity, whose mean value P is determined by the saturable gain function U [see Eq. (11)]. The relation between the thermodynamics in the fixed-power ensemble analyzed above, and the variable-power ensemble which is the subject of this section is quite similar to the one between the canonical and grand canonical ensembles in statistical mechanics [23]. In the latter case one defines the grand potential $\Omega = \mu N - F$, where $\mu = \partial F / \partial N$ is the chemical potential. An equivalent thermodynamics is obtained after replacing the extensive variable N , by the intensive variable μ . Finite size corrections to the thermodynamics in the two ensembles are also related, but not equivalent.

In this section we show in a similar spirit that thermodynamics with fixed and variable power is equivalent, and calculate the thermodynamic limit of $P \equiv \langle \mathcal{P} \rangle$ in the variable power case. It is quite straightforward to generalize the calculations and to obtain subleading terms as in Sec. IV, but this is not pursued here.

In the fixed power ensemble the free energy per degree of freedom \tilde{F} of the Hamiltonian \tilde{H}_N which includes the saturable gain is related to the free energy F calculated in Sec. IV by

$$\tilde{F}(\gamma_s, T, P) = F(\gamma_s P^2/T) + \log P + u(P). \quad (55)$$

[Refer to Eqs. (11) and (15) for the relevant definitions.] We now define the variable-power thermodynamic potential Φ as the Legendre transform of \tilde{F} ,

$$\Phi(\gamma_s, T, \mu) = \min_P \mu P - \tilde{F}(\gamma_s, T, P). \quad (56)$$

The function u has to grow faster than P^2 to ensure that the minimum in Eq. (56) is finite. We also require that u is convex to obtain a unique minimum. Otherwise u is arbitrary.

The thermodynamics is obtained from Φ through the properties of the Legendre transform, the case of interest being $\mu=0$. The mean power,

$$P = \partial_{\mu} \Phi(\gamma_s, T, 0), \quad (57)$$

can be found from the definition Eq. (56) and the results of the preceding section; it is given implicitly by

$$1 + Pu'(P) - \gamma \bar{y}(\gamma)^2 = 0, \quad (58)$$

where as before $\gamma = \gamma_s P^2 / T$. The order parameter is

$$M^4 = 2T \partial_{\gamma_s} \Phi = -2T \partial_{\gamma_s} \tilde{F} = -P^2 F'(\gamma) = P^2 \bar{y}(\gamma)^2, \quad (59)$$

which, for a given mean power P , is *independent* of the form of u , and therefore also equal to the order parameter in the fixed power ensemble. Moreover, the thermodynamics depends on the single parameter γ . A special case is the mode-locking transition point, which occurs at $\gamma = \gamma^*$ whatever the form of the saturable gain function. This universal behavior stems from the thermodynamic equivalence of the fixed power and variable power ensembles.

Another interesting thermodynamic quantity which can be studied only in the variable power framework is the susceptibility $\chi = P'(\gamma)$ which measures the response of the intracavity power to changes in the strength of the nonlinearity or inverse noise power. Taking the derivative of Eq. (58) shows that

$$\chi = \frac{[\gamma \bar{y}(\gamma)^2]'}{[Pu'(P)]'}. \quad (60)$$

In the non-mode-locked regime P is independent of γ and $\chi = 0$; when $\bar{y} > 0$ and mode-locking occurs, $\bar{y}'(\gamma)$ is also positive and it follows from the convexity of u that the susceptibility is strictly positive in mode-locked systems.

ACKNOWLEDGMENTS

We are pleased to acknowledge fruitful discussions with Shmuel Fishman. This work was supported by the Israeli Science Foundation (ISF) founded by the Israeli Academy of Sciences.

-
- [1] H. W. Mocker and R. J. Collins, *Appl. Phys. Lett.* **7**, 270 (1965).
 [2] H. Haken and H. Ohno, *Opt. Commun.* **16**, 205 (1976).
 [3] H. Haken, *Synergetics*, 2nd enlarged ed. (Springer-Verlag, Berlin, 1978).
 [4] G. H. C. New, *Proc. IEEE* **67**, 380 (1979).
 [5] F. Krausz, T. Brabec, and Ch. Spillmann, *Opt. Lett.* **16**, 235 (1991).
 [6] H. A. Haus and E. P. Ippen, *Opt. Lett.* **16**, 1331 (1991).
 [7] J. Herrmann, *Opt. Commun.* **98**, 111 (1993).
 [8] K. Tamura, J. Jacobson, E. P. Ippen, H. A. Haus, and J. G. Fujimoto, *Opt. Lett.* **18**, 220 (1993).
 [9] F. Krausz and T. Brabec, *Opt. Lett.* **18**, 888 (1993).
 [10] Y.-F. Chou, J. Wang, H.-H. Liu, and N.-P. Kuo, *Opt. Lett.* **19**, 566 (1994).
 [11] C. J. Chen, P. K. A. Wai, and C. R. Menyuk, *Opt. Lett.* **20**, 350 (1995).
 [12] T. Kapitula, J. N. Kutz, and B. Sandstede, *J. Opt. Soc. Am. B* **19**, 740 (2002).
 [13] A. Gordon and B. Fischer, *Opt. Lett.* **18**, 1326 (2003).
 [14] A. Gordon and B. Fischer, *Phys. Rev. Lett.* **89**, 103901 (2002).
 [15] A. Gordon and B. Fischer, *Opt. Commun.* **223**, 151 (2003).
 [16] A. Gordon *et al.*, Proceedings of the Conference on Lasers and Electro-Optics, Baltimore, 2004, session CWM6 (unpublished).
 [17] H. A. Haus, *IEEE J. Sel. Top. Quantum Electron.* **6**, 1173 (2000).
 [18] A. E. Siegman, *Lasers* (University of Science Books, Mill Valley, CA, 1986).
 [19] C. J. Chen, P. K. Wai, and C. R. Menyuk, *Opt. Lett.* **19**, 198 (1994).
 [20] T. H. Berlin and M. Kac, *Phys. Rev.* **86**, 821 (1952).
 [21] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag, New York, 1989).
 [22] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Clarendon, Oxford, 1996).
 [23] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 3rd ed. (Pergamon, Oxford, 1980).